SEC-9.1 LU-DECOMPOSITIONS.

Upto now, we have focused on two methods for solving linear systems, Gaussian elimination (reduction to row echelon form) and Gauss-Jordan elimination (reduction to reduced row echelon form). While these methods are fine for small-scale problems in this text, they are not suitable for large-scale problems in which computer round-off error, memory usage, and speed are concerns. In this Section, we will discuss a method for solving a linear system of n-equations in n-unknowns that is based on factoring its coefficient matrix into a product of lower and upper triangular matrices. This method, called "LU-Decomposition" is the basis for many computer algorithms in common use.

SOLVING LINEAR SYSTEMS BY FACTORING

Our first goal in this Section is to show how to solve a linear system $A\mathbf{x} = \mathbf{b}$ of n equations in n unknowns by factoring the coefficient matrix A into a product

$$A = LU \qquad \text{—— (1)}$$

where L is lower triangular and U is upper triangular matrix.

Once we understand how to do this, we will discuss how to obtain the factorization itself.

The method of LU-Decomposition

Step (1) Rewrite the system $A\mathbf{x} = \mathbf{b}$ as

$$LU\mathbf{x} = \mathbf{b} \qquad \text{—— (2)}$$

Step (2) Define a new $n \times 1$ matrix by

$$U\mathbf{x} = \mathbf{y} \qquad \text{—— (3)}$$

Step (3) Use (3) to rewrite (2) as

$$L\mathbf{y} = \mathbf{b} \qquad \text{—— (4)}$$

and solve this system for $\mathbf{y}$.

Step (4) Substitute $\mathbf{y}$ in (3) and solve for $\mathbf{x}$.

Example (1) Solving $A\mathbf{x} = \mathbf{b}$ by LU-Decomposition

Solve the linear system

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 \quad &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned} \quad \text{OR} \quad \underbrace{\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

using factorization

$$\underbrace{\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Solution : The given system of equations is

$$Ax = b \quad \text{———} \quad (1) \quad , \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , \; b = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Using factorization , $A = LU$ in (1), we get

$$LUx = b \quad \text{———} \quad (2)$$

Now let $Ux = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ — (3)

Using (3), the equ. (2) becomes

$$LY = b$$

or

$$\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$, using factorization of A

or equivalently

$$\left.\begin{aligned} 2y_1 \qquad\quad &= 2 \\ -3y_1 + y_2 \quad &= 2 \\ 4y_1 - 3y_2 + 7y_3 &= 3 \end{aligned}\right\} \quad \text{———} \quad (4)$$

Solving the system of equ. (4) by forward substitution, we get

$$y_1 = 1 , \; y_2 = 5 , \; y_3 = 2$$

Substituting these values into (3) yields the linear system

Ux=Y

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

or equivalently

$$\left.\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ x_3 &= 2 \end{aligned}\right\} \quad \text{—} \quad (5)$$

Solving the system of equ. (5) by back substitution yields

$x_1 = 2$, $x_2 = -1$, $x_3 = 2$, which is the soln. of given system of equ.

FINDING LU-DECOMPOSITIONS

Definition: A factorization of a square matrix 'A' as $A = LU$, where L is lower triangular and U is upper triangular matrix is called an LU-Decomposition (or LU-factorization) of 'A'.

NOTE: Not every square matrix has an LU-Decomposition. However, we will see that if it is possible to reduce a square matrix 'A' to row echelon form by Gaussian elimination without performing any row interchanges, then matrix 'A' will have an LU-decomposition, though it may not be unique.

To see why this is so, assume that 'A' has been reduced to a row-echelon form U using a sequence of row operations that does not include row interchanges. We know from a Theorem of Sec 1.5 that these operations can be accomplished by multiplying 'A' on the left by an appropriate sequence of elementary matrices; that is, there exists elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$E_k \cdots E_2 E_1 A = U \qquad \text{——— (1)}$$

Since elementary matrices are invertible, we can solve (1) for 'A' as

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

or more briefly as

$$A = LU \qquad \text{——— (2)}$$

where

$$L = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \qquad \text{——— (3)}$$

THEOREM (1) If 'A' is a square matrix that can be reduced to a row echelon form U by Gaussian elimination without row interchanges, then 'A' can be factored as

$A = LU$, where L is lower triangular matrix.

Example (2) An LU-Decomposition

Find an LU-decomposition of
$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

PROCEDURE FOR CONSTRUCTING AN LU-DECOMPOSITION

Step ① Reduce 'A' to row echelon form U by Gaussian elimination without row interchanges, keeping track of the multipliers used to introduce the leading 1's and the multipliers used to introduce the zeros below the leading 1's.

Step ② In each position along the main diagonal of L, place the reciprocal of the multiplier that introduced the leading 1 in that position in U.

Step ③ In each position below the main diagonal of L, place the negative of the multiplier used to introduce the zero in that position in U.

Step ④ Form the decomposition $A = LU$.

$\boxed{\text{LU-decompositions are not unique.}}$

Example ③ Constructing an LU-Decomposition

Find an LU-decomposition of the matrix, $A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$.

Soln. We will reduce 'A' to a row echelon form U and at each step, we will fill in an entry of L in accordance with the four-step procedure above —

$\boxed{\bullet \text{ denotes an unknown entry of } L}$

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \leftarrow \text{multiplier} = \frac{1}{6} \qquad \begin{bmatrix} 6 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = -9 \\ \leftarrow \text{multiplier} = -3 \end{matrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & \bullet & 0 \\ 3 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{bmatrix} \leftarrow \text{multiplier} = \frac{1}{2} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = -8 \end{matrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = 1 \end{matrix} \qquad L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

Thus, we have constructed the LU-decomposition

$$A = LU = \cancel{\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

We can confirm this end result by multiplying the factors.

SEC (9.2) THE POWER METHOD

The Eigenvalues of a square matrix can, in theory, be found by solving the characteristic equation. However, this procedure has so many computational difficulties that it is almost never used in applications. In this Section, we will discuss an algorithm that can be used to approximate the eigenvalue with greatest absolute value and a corresponding eigenvector. This particular eigenvalue and its corresponding eigenvectors are important because they arise naturally in many iterative processes. The methods we will study in this Section have recently been used to create Internet search engines such as Google.

THE POWER METHOD – There are many applications in which some vector x_0 in R^n is multiplied repeated by an $n \times n$ matrix 'A' to produce a sequence $x_0, Ax_0, A^2x_0, \ldots, A^kx_0, \ldots$

We call a sequence of this form a power sequence generated by A. In this Section, we will be concerned with the convergence of power sequences and how such sequences can be used to approximate eigenvalues and eigenvectors. For this purpose, we make the following defi:–

Definition – If the distinct eigenvalues of a matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and if $|\lambda_1|$ is larger than $|\lambda_2|, \ldots, |\lambda_k|$ then λ_1 is called a Dominant Eigenvalue of 'A'. Any eigenvector corresponding to a dominant eigenvalue is called a Dominant Eigenvector of 'A'.

Example (1) Dominant Eigenvalues

Some matrices have dominant eigenvalues and some do not.

For example, if the distinct eigenvalues of a matrix are $\lambda_1 = -4, \lambda_2 = -2, \lambda_3 = 1, \lambda_4 = 3$ then $\lambda_1 = -4$ is dominant since $|\lambda_1| = 4$ is greater than the absolute values of all the other eigenvalues;

But if the distinct eigenvalues of a matrix are $\lambda_1 = 7, \lambda_2 = -7, \lambda_3 = -2, \lambda_4 = 5$ then $|\lambda_1| = |\lambda_2| = 7$ so there is no eigenvalue whose absolute value is greater than the absolute value of all the other eigenvalues.

THEOREM (1) Let 'A' be a Symmetric $n \times n$ matrix with a positive dominant eigenvalue λ. If x_0 is a unit vector in R^n that is not orthogonal to the eigenspace corresponding to λ, then the normalized power sequence

$$x_0, \quad x_1 = \frac{Ax_0}{\|Ax_0\|}, \quad x_2 = \frac{Ax_1}{\|Ax_1\|}, \quad \ldots, \quad x_k = \frac{Ax_{k-1}}{\|Ax_{k-1}\|}, \ldots \quad — (1)$$

converges to a unit dominant eigenvector, and the sequence

$$Ax_1 \cdot x_1, \quad Ax_2 \cdot x_2, \quad \ldots, \quad Ax_k \cdot x_k, \ldots \quad — (2)$$

converges to the dominant eigenvalue λ.

THE POWER METHOD WITH EUCLIDEAN SCALING

Theorem ① provides us with an algorithm for approximating the dominant eigenvalue and a corresponding unit eigenvector of a symmetric matrix A, provided the dominant eigenvalue is positive. This algorithm, called the Power Method with Euclidean Scaling, is as follows —

Step ① Choose an arbitrary non-zero vector and normalize it, if need be, to obtain a unit vector x_0.

Step ② Compute Ax_0 and normalize it to obtain the first approximation x_1 to a dominant unit eigenvector. Compute $Ax_1 \cdot x_1$ to obtain the first approximation to the dominant eigenvalue.

Step ③ Compute Ax_1 and normalize it to obtain the second approximation x_2 to a dominant unit eigenvector. Compute $Ax_2 \cdot x_2$ to obtain the second approximation to the dominant eigenvalue.

Continuing in this way will usually generate a sequence of better and better approximations to the dominant eigenvalue and a corresponding unit eigenvector.

Example ② The Power Method with Euclidean Scaling

Apply Power method with Euclidean scaling to $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Stop at x_5 and compare resulting approximations to exact values of dominant eigenvalue & eigenvector.

Soln. The Eigenvalues of 'A' are $\lambda = 1$ and $\lambda = 5$

The eigenspace corresponding to dominant eigenvalue $\lambda = 5$ is $x = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ —①

Setting $t = \frac{1}{\sqrt{2}}$ yields the normalized dominant eigenvector $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 0.70710678\ldots \\ 0.70710678\ldots \end{bmatrix}$ —②

POWER METHOD

$$Ax_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad x_1 = \frac{Ax_0}{\|Ax_0\|} = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \frac{1}{3.60555}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix}$$

$$Ax_1 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix} \approx \begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix}, \quad x_2 = \frac{Ax_1}{\|Ax_1\|} \approx \frac{1}{4.90682}\begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix} \approx \begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix}$$

$$Ax_2 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx \begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix}, \quad x_3 = \frac{Ax_2}{\|Ax_2\|} \approx \frac{1}{4.99616}\begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix} \approx \begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix}$$

$$Ax_3 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx \begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix}, \quad x_4 = \frac{Ax_3}{\|Ax_3\|} \approx \frac{1}{4.99985}\begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix} \approx \begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix}$$

$$Ax_4 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx \begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix}, \quad x_5 = \frac{Ax_4}{\|Ax_4\|} \approx \frac{1}{4.99999}\begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix} \approx \begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix}$$

$$\lambda^{(1)} = (Ax_1)\cdot x_1 = (Ax_1)^T x_1 \approx \begin{bmatrix} 3.60555 & 3.32820 \end{bmatrix}\begin{bmatrix} 0.80205 \\ 0.55470 \end{bmatrix} \approx 4.84615$$

$$\lambda^{(2)} = (Ax_2)\cdot x_2 = (Ax_2)^T x_2 \approx \begin{bmatrix} 3.56097 & 3.50445 \end{bmatrix}\begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx 4.99361$$

$$\lambda^{(3)} = (Ax_3)\cdot x_3 = (Ax_3)^T x_3 \approx \begin{bmatrix} 3.54108 & 3.52976 \end{bmatrix}\begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx 4.99974$$

$$\lambda^{(4)} = (Ax_4)\cdot x_4 = (Ax_4)^T x_4 \approx \begin{bmatrix} 3.53666 & 3.53440 \end{bmatrix}\begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx 4.99999$$

$$\lambda^{(5)} = (Ax_5)\cdot x_5 = (Ax_5)^T x_5 \approx \begin{bmatrix} 3.53576 & 3.53531 \end{bmatrix}\begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix} \approx 5.00000$$

SEC (9.5) SINGULAR VALUE DECOMPOSITION

In this Section, we will discuss an extension of the diagonalization theory for $n\times n$ symmetric matrices to general $m\times n$ matrices. The results that we will develop in this Section have applications to compression, storage and transmittion of digitized information and form the basis for many of the best computational algorithms that are currently available for solving linear systems.

SINGULAR VALUES. Since matrix products of the form A^TA will play an important role in our work, we will begin with two basic theorems about them.

THEOREM 1 If 'A' is an $m\times n$ matrix, then

(i) 'A' and 'A^TA' have the same null space.

(ii) 'A' and 'A^TA' have the same row space.

(iii) 'A' and 'A^TA' have the same column space.

(iv) 'A' and 'A^TA' have the same rank.

THEOREM 2 If 'A' is an $m\times n$ matrix, then

(i) 'A^TA' is orthogonally diagonalizable.

(ii) The eigenvalues of 'A^TA' are non-negative.

Definition: If 'A' is an $m\times n$ matrix, and if $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigenvalues of 'A^TA', then the numbers $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\cdots$, $\sigma_n = \sqrt{\lambda_n}$ are called Singular Values of 'A'.

Example 1 Find the Singular Values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: The first step is to find the eigenvalues of the matrix A^TA.

Here $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic ~~equation~~ equ. of A^TA is $\det(\lambda I - A^TA) = 0$

i.e., $\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 0$

$\Rightarrow \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0$

$\Rightarrow (\lambda-2)(\lambda-2) - (-1)(-1) = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow (\lambda-3)(\lambda-1) = 0 \Rightarrow \lambda = 3, 1$

So the eigenvalues of A^TA are $\lambda_1 = 3$ & $\lambda_2 = 1$.

and singular values of 'A' in order of decreasing size are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$; $\sigma_2 = \sqrt{\lambda_2} = 1$.

Singular Value Decomposition :

Before turning to the main result in this Section, we will find it useful to extend the notion of a 'main diagonal' to matrices that are not square. We define the main diagonal of an mxn matrix to be the line of entries shown in Fig. – it starts at the upper left corner and extends diagonally as far as it can go. We will refer to the entries on main diagonal as diagonal entries.

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix} , \quad \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

We are now ready to consider the main result in this Section, which is concerned with a specific way of factoring a general mxn matrix A. This factorization

SEC (10.2) GEOMETRIC LINEAR PROGRAMMING

In this Section, we describe a geometric technique for maximizing or minimizing a linear expression in two variables subject to a set of linear constraints. Let us begin with some examples —

Example (1) Maximizing Sales Revenue

A candy manufacturer has 130 pounds of chocolate-covered cherries and 170 pounds of chocolate-covered mints in stock. He decides to sell them in the form of two different mixtures. One mixture will contain half cherries and half mints by weight and will sell for \$2.00 per pound. The other mixture will contain one-third cherries and two-thirds mints by weight and will sell for \$1.25 per pound. How many pounds of each mixture should the candy manufacturer prepare in order to maximize his sales revenue?

Mathematical Formulation

Let the mixture of half cherries and half mints be called mix A, and let x_1 be the no. of pounds of this mixture to be prepared.

Let the mixture of one-third cherries and two-thirds mints be called mix B and let x_2 be the no. of pounds of this mixture to be prepared.

Since mix A sells for \$2.00 per pound and mix B sells for \$1.25 per pound, the total sales z (in dollars) will be

$$z = 2.00x_1 + 1.25x_2$$

Since each pound of mix A contains $\frac{1}{2}$ pound of cherries and each pound of mix B contains $\frac{1}{3}$ pound of cherries, the total no. of pounds of cherries used in both mixtures is

$$\frac{1}{2}x_1 + \frac{1}{3}x_2$$

Similarly, since each pound of mix A contains $\frac{1}{2}$ pound of mints and each pound of mix B contains $\frac{2}{3}$ pound of mints, the total no. of pounds of mints used in both mixtures is

$$\frac{1}{2}x_1 + \frac{2}{3}x_2$$

Because the manufacturer can use at most 130 pounds of cherries and 170 pounds of mints, we must have

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \le 130$$

$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \le 170$$

Furthermore, since x_1 & x_2 cannot be negative numbers, we must have $x_1 \geq 0$ and $x_2 \geq 0$.

The problem can therefore be formulated mathematically as follows —

Find values of x_1 & x_2 that maximize $z = 2.00x_1 + 1.25x_2$

subject to

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \le 130$$

$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \le 170$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

GEOMETRIC SOLUTION OF LINEAR PROGRAMMING PROBLEMS

Each of the preceding examples is a special case of the following problem —

Problem: Find values of x_1 & x_2 that either maximize or minimize

$$z = c_1 x_1 + c_2 x_2 \quad \text{— ①} \qquad \text{[Objective Function]}$$

Linear

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 \ (\le, \ge, \text{or} =) \ b_1 \\ a_{21}x_1 + a_{22}x_2 \ (\le, \ge, \text{or} =) \ b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 \ (\le, \ge, \text{or} =) \ b_m \end{array}\right\} \quad \text{— ②} \quad \text{[Constraints]}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0 \quad \text{— ③} \quad \text{[Non-Negativity Constraints]}$$

The problem above is called General Linear Programming Problem in two variables. The linear function **z** in ① is called the Objective Function. Equ. ② & ③ are called Constraints; in particular, equ. in ③ are called Non-Negativity Constraints on variables x_1 & x_2.

We will now show how to solve a linear programming problem in two variables graphically. A pair of values (x_1, x_2) that satisfy all of the constraints is called a Feasible Solution. The set of all feasible solutions determines a subset of x_1x_2-plane called the Feasible Region. Our desire is to find a feasible solution that maximizes the objective function. Such a solution is called an Optimal Solution.

To examine the feasible region of a linear programming problem, let us note that each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 = b_i$$

defines a line in the x_1x_2-plane, whereas each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 \le b_i \quad \text{OR} \quad a_{i1}x_1 + a_{i2}x_2 \ge b_i$$

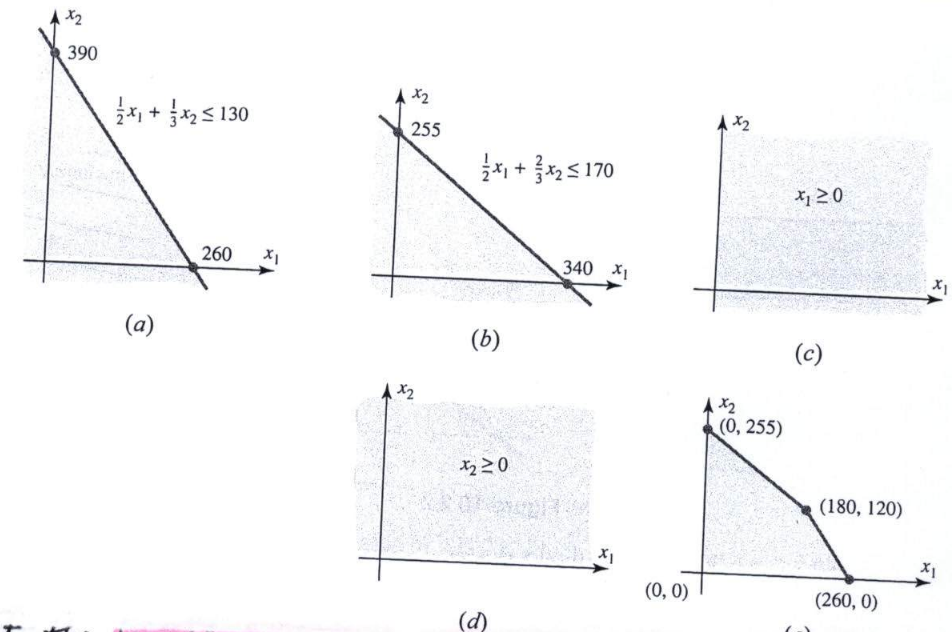
defines a half-plane that includes its boundary line $a_{i1}x_1 + a_{i2}x_2 = b_i$

Thus, the feasible region is always an intersection of finitely many lines and half planes.

For example, the four constraints

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \le 130$$
$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \le 170$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

of Example ① define the half planes illustrated in parts (a), (b), (c) & (d) of Fig. The feasible region of this problem is thus the intersection of these four half planes, which is illustrated in Fig (e).



It can be shown that the feasible region of a linear programming problem has a boundary consisting of a finite no. of straight line segments. If the feasible region can be enclosed in a sufficiently large circle, it is called Bounded; otherwise it is called Unbounded. If the feasible region is empty (contains no points), then the constraints are inconsistent and the linear programming problem has no solution.

Those boundary points of a feasible region that are intersections of two of the straight line boundary segments are called Extreme Points (They are also called Corner Points and Vertex Points).

The importance of the extreme points of a feasible region is shown by the following theorem —

THEOREM ① Maximum and Minimum Values

If the feasible region of a linear programming problem is non-empty and bounded, then the objective function attains both a maximum and a minimum value and these occur at extreme points of the feasible region. If the feasible region is unbounded, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at an extreme point.

Example ④ Example ① Revisited Using Theorem ①

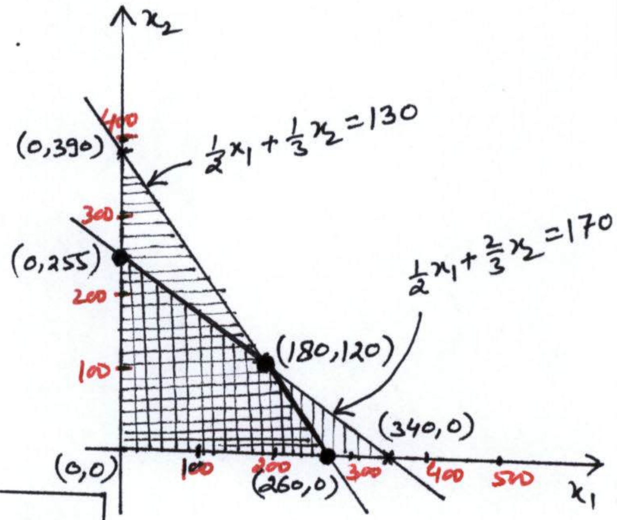
Find values of x_1 & x_2 that maximize $z = 2.00x_1 + 1.25x_2$

Subject to

$$\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 \le 130$$
$$\tfrac{1}{2}x_1 + \tfrac{2}{3}x_2 \le 170$$

and $x_1 \ge 0$, $x_2 \ge 0$.

Solu. In Fig., we have drawn the feasible region of this problem. Since, it is bounded, the maximum value of objective function (z) is attained at one of the four extreme points. The values of the objective function at the four extreme points are given in the following Table —



Extreme Point (x_1, x_2)	(0,0)	(0,255)	(180,120)	(260,0)
Value of $z = 2x_1 + 1.25x_2$	0	318.75	510.00	520.00

We see that the max. value of z is 520.00 and the corresponding optimal solu. is (260, 0). Thus, the candy manufacturer attains maximum sales of \$520 when he produces 260 pounds of mixture A and none of mixture B.

Example ⑤ Using Theorem ①

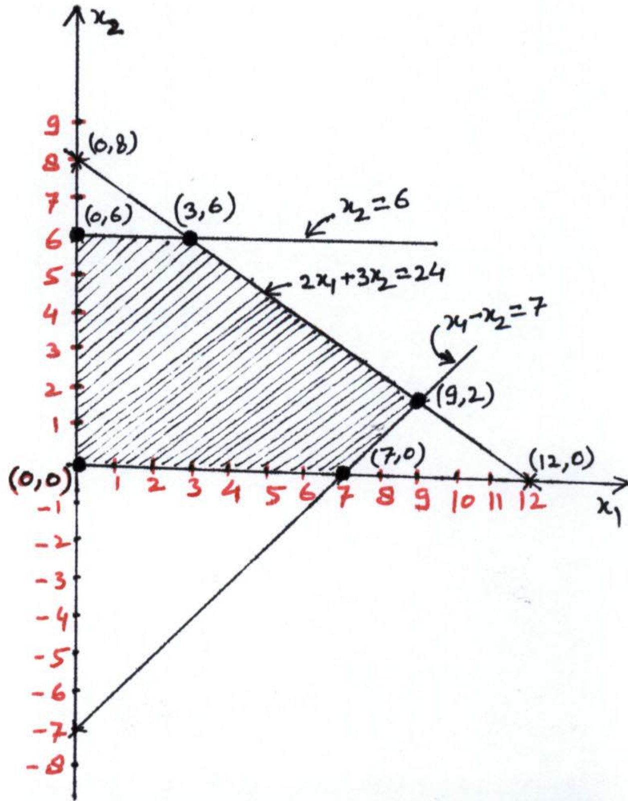
Find the values x_1 & x_2 that maximize $z = x_1 + 3x_2$

subject to

$$2x_1 + 3x_2 \le 24$$
$$x_1 - x_2 \le 7$$
$$x_2 \le 6$$

and $x_1 \ge 0$, $x_2 \ge 0$.

Solu. In Fig., we have drawn the feasible region of this problem. Since, it is bounded, the max. value of the objective function (z) is attained at one of the five extreme points. The values of the objective function at the five extreme points are given in the following Table —



Extreme Point	(0,0)	(0,6)	(3,6)	(9,2)	(7,0)
Value of $z = x_1 + 3x_2$	0	18	21	15	7

From this Table, the max. value of $z = 21$, which is attained at $x_1 = 3$ and $x_2 = 6$.

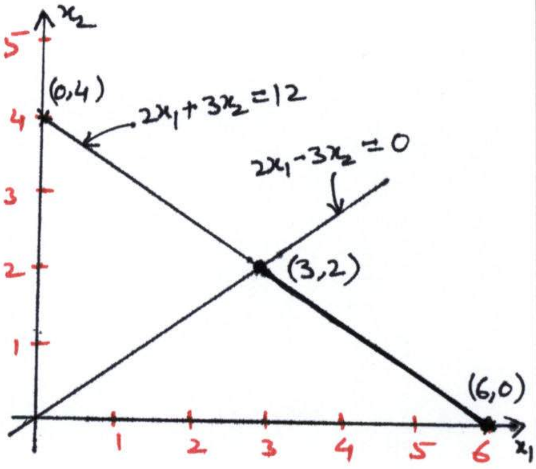
Example ⑥ The Feasible Region Is a Line Segment

Find the values of x_1 & x_2 that minimize $z = 2x_1 - x_2$

subject to $2x_1 + 3x_2 = 12$

$2x_1 - 3x_2 \geq 0$

$x_1 \geq 0, \; x_2 \geq 0$.



Solu. In fig., we have drawn the feasible region of this problem. Because one of the ~~cont~~ constraints is an equality constraint, the feasible region is a straight line with two extreme points.

The values of z at the two extreme points are given in the following Table —

Extreme Point (x_1, x_2)	(3,2)	(6,0)
Value of $z = 2x_1 - x_2$	4	12

Thus, the minimum value of z is 4 and is attained at $x_1 = 3$ & $x_2 = 2$.

Example ⑦ Unbounded Solution

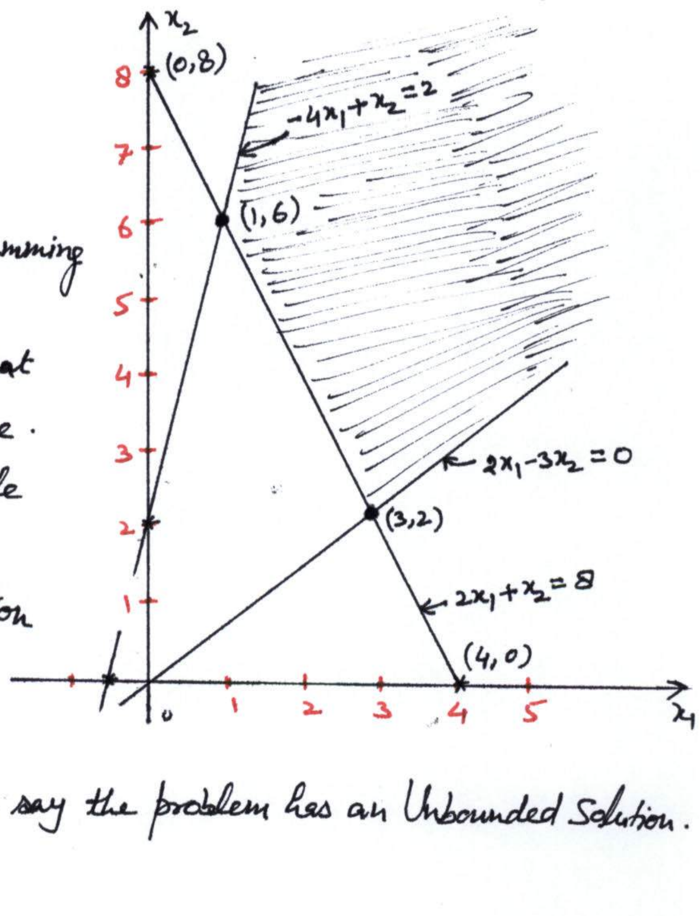
Find the values of x_1 & x_2 that maximize $z = 2x_1 + 5x_2$

subject to $2x_1 + x_2 \geq 8$

$-4x_1 + x_2 \leq 2$

$2x_1 - 3x_2 \leq 0$

$x_1 \geq 0, \; x_2 \geq 0$.



Solu. The feasible region of this linear programming problem is illustrated in fig. Since the region is unbounded, we are not assured by Theorem ① that the objective function attains a maximum value. In fact, it is easily seen that since the feasible region contains points for which both x_1 & x_2 are arbitrarily large and positive, the objective function

$$z = 2x_1 + 5x_2$$

can be made arbitrarily large and positive.

This problem has no Optimal Solu. Instead we say the problem has an Unbounded Solution.

Example (8)

Find values of x_1 & x_2 that maximize $z = -5x_1 + x_2$

subject to
$$2x_1 + x_2 \geq 8$$
$$-4x_1 + x_2 \leq 2$$
$$2x_1 - 3x_2 \leq 0$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Soln.

The above constraints are the same as those in Example (7), so the feasible region of this problem is also given by the fig. given in Example (7). The objective function of this problem attains a maximum within the feasible region. By Theorem (1), this maximum must be attained at an extreme point. The values of z at the two extreme points of the feasible region are given in the following Table —

Extreme Point (x_1, x_2)	(1,6)	(3,2)
Value of $z = -5x_1 + x_2$	1	-13

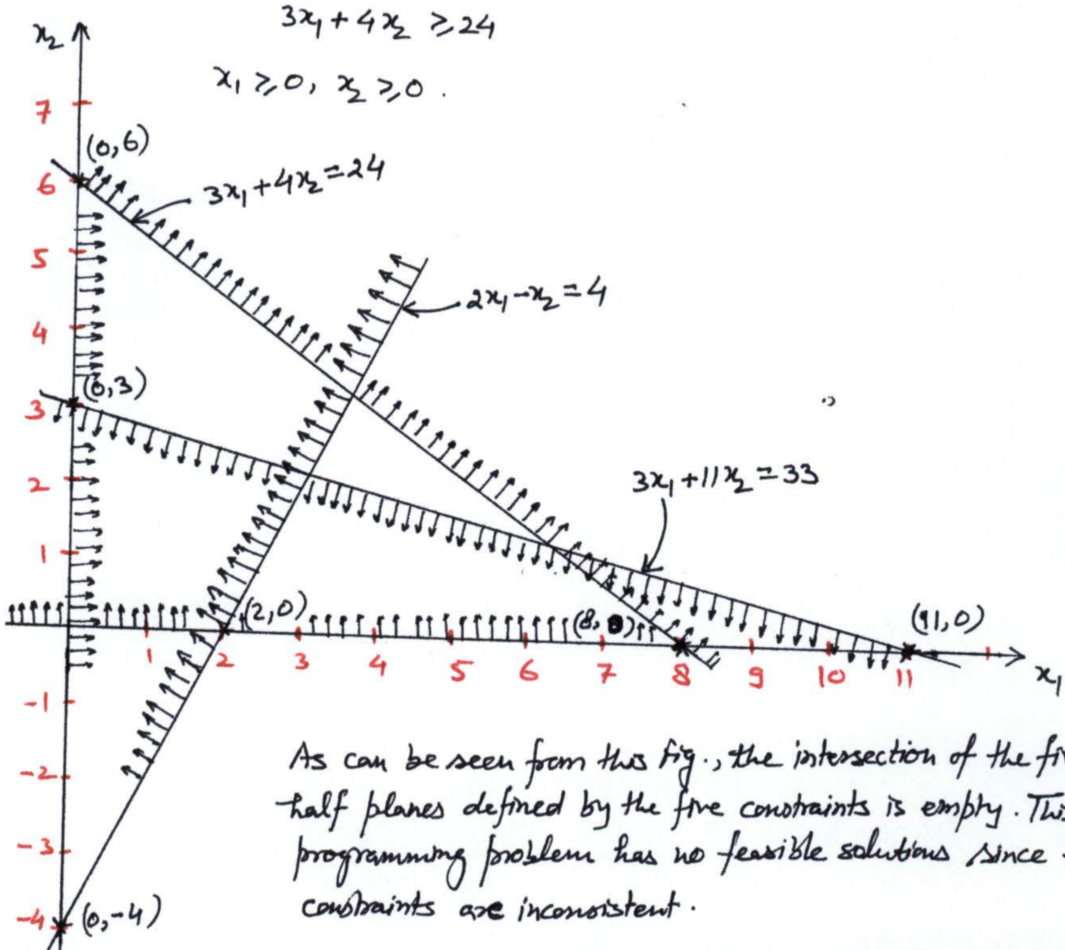
Thus, the maximum value of z is 1 and is attained at extreme point $x_1 = 1, x_2 = 6$.

Example (9) Inconsistent Constraints

Find the values of x_1 & x_2 that minimize $z = 3x_1 - 8x_2$

subject to
$$2x_1 - x_2 \leq 4$$
$$3x_1 + 11x_2 \leq 33$$
$$3x_1 + 4x_2 \geq 24$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Soln.



As can be seen from this fig., the intersection of the five half planes defined by the five constraints is empty. This linear programming problem has no feasible solutions since the constraints are inconsistent.